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Problems on polygons and Bonnesen-type inequalities [1]

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Abstract

In this paper we are interested in some Bonnesen-type isoperimetric inequalities for plane n-gons in relation with the two conjectures proposed by P. Levy and X.M. Zhang.

1 Introduction

As a well known result, for a simple closed curve $\mathcal{C}$ (in the euclidian plane) of length L enclosing a domain of area A, we have the inequality

$$L^2 - 4\pi A \geq 0. \tag{1}$$

Equality is attained if and only if this curve is a euclidean circle. This means that among the set of domains of fixed area, the euclidean circle has the smallest perimeter.

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The above inequality (1) could be easily deduced from the Wirtinger inequality

$$\int_0^{2\pi} |f'(x)|^2 dx \geq \int_0^{2\pi} |f(x)|^2 dx, \tag{2}$$

where $f(x)$ is a continuous periodic function of period 2π whose derivative $f'(x)$ is also continuous and $\int_0^{2\pi} f(x)dx = 0$. Equality holds if and only if $f(x) = \alpha \cos x + \beta \sin x$ (see Osserman's paper [O1]).
For any curve $\mathcal{C}$ of length L enclosing an area A, the quantity $L^2 - 4\pi A$ is called *the isoperimetric deficiency* of $\mathcal{C}$, because it decreases towards zero when $\mathcal{C}$ tends to a circle.
As an extension, Bonnesen proves [O1] that if $\mathcal{C}$ is convex and there exists a circular annulus containing $\mathcal{C}$ of thickness d, then we have

$$4\pi d^2 \leq L^2 - 4\pi A.$$

In fact, Fuglede showed that convexity is not a necessary hypothesis [F].

There is a related isoperimetric inequality known as the Bonnesen :

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{3}$$

where R is the circumradius and r is the inradius of the curve $\mathcal{C}$.
Note that if the right side of (3) equals zero, then $R = r$. This means that $\mathcal{C}$ is a circle and $L^2 - 4\pi A = 0$.
More generally, inequalities of the form

$$L^2 - 4\pi A \geq K, \tag{4}$$

are called Bonnesen-type isoperimetric inequalities if equality is only attained for the euclidean circle. In the other words, K is positive and satisfies the condition

$$K = 0 \quad \text{implies} \quad L^2 - 4\pi A = 0.$$

(See [O2] for a general discussion and different generalisations).

For an n-gon Π_n (a polygon with n sides) of perimeter L_n and area A_n, the following inequality is known

$$L_n^2 - 4(n\tan\frac{\pi}{n})A_n \geq 0. \qquad (5)$$

Equality is attained if and only if the n-gon is regular. Thus, if we consider a smooth curve as a polygon with infinitely many sides, it appears that inequality (1) is a limiting case of (5).

Moreover, we know that an n-gon has a maximum area among all n-gons with the given set of sides if it is convex and inscribed in a circle. Let $a_1, a_2, ..., a_n$ denote the lengths of the sides of Π_n. For a triangle, Heron's formula gives the area

$$A_3 = \frac{1}{4}L_3^2\sqrt{(1-\frac{2a_1}{L_3})(1-\frac{2a_2}{L_3})(1-\frac{2a_3}{L_3})}.$$

For a quadrilateral, the Brahmagupta formula gives a bound for the area

$$A_4 \leq \frac{1}{4}L_4^2\sqrt{(1-\frac{2a_1}{L_4})(1-\frac{2a_2}{L_4})(1-\frac{2a_3}{L_4})(1-\frac{2a_4}{L_4})}$$

with equality if and only if Π_4 can be inscribed in a circle.

2 Isoperimetric constants

We can ask if it is possible to get an analogous formula for other plane polygons (not necessarilly inscribed in a circle). More precisely, is the area A_n of the n-gon is close to the following expression ?

$$P_n = \frac{L_n^2}{4}\sqrt{(1-\frac{2a_1}{L_n})(1-\frac{2a_2}{L_n})(1-\frac{2a_3}{L_n}).....(1-\frac{2a_n}{L_n})} \qquad (6)$$

This question has been considered by many geometers who tried to compare A_n with P_n. One of them, P. Levy [L] was interested in this problem and more precisely he expected the following

Conjecture 1 : *Define the ratio* $\varphi_n = \frac{A_n}{P_n}$. *For any n-gon* Π_n, *with sides* $a_1, a_2, ..., a_n$ *enclosing an area* A_n, *and* P_n *defined as above, this ratio verifies*

$$a) \quad \frac{e}{\pi} \leq \varphi_n \qquad and \qquad b) \quad \varphi_n \ \leq \ 1.$$

Remark : *Notice that part a) is obviously only valid for cyclic n-gons, but part b) of Conjecture* 1 *may be true for any n-gon. In particular, for a triangle we have* $\varphi_3 = 1$ *and for a quadrilateral,* $\varphi_4 \leq 1$ ($\varphi_4 = 1$ *in the cyclic case).*

Conjecture 1 was originally motivated by study of cyclic n-gons. More precisely, for regular n-gons we get $a_i = L_n/n$. The associated value of φ_n is given by

$$\varphi_n{}^0 = \frac{1}{n \tan\frac{\pi}{n}\ (1 - \frac{2}{n})^{n/2}} \tag{7}$$

and satisfies the inequalities of Conjecture 1. Moreover, we may verify that ${\varphi^0}_n$ is a decreasing function in n.

As we shall see below, the lower bound $\varphi_n \geq \frac{e}{\pi}$ seems to have more geometric interest than the upper one. Indeed, it allows one to estimate the defect between any n-gon Π_n and the regular one. This defect may be measured by the quotient

$$\tau_n = \frac{\varphi_n}{\varphi_n{}^0} \tag{8}$$

which tends to 1 whenever Π_n is close to being regular. Moreover, τ_n is related to a new Bonnesen-type inequality for plane polygons.

On the other hand, H.T. Ku, M.C. Ku and X.M. Zhang, ([K.K.Z] and [Z]) have been interested in this same problem. Their approach is quite different. They consider the so called pseudo-perimeter of second kind $\hat{L}_n$ defined by

$$\hat{L}_n = L_n(\frac{n}{n-2})[(1-\frac{2a_1}{L_n})(1-\frac{2a_2}{L_n})(1-\frac{2a_3}{L_n}).....(1-\frac{2a_n}{L_n})]^{1/n}. \tag{9}$$

In fact, there is a relation between $\hat{L}_n$ and P_n

$$\hat{L}_n = (\frac{n}{n-2})(4P_n)^{\frac{2}{n}} {L_n}^{\frac{n-4}{n}}. \tag{10}$$

X.M. Zhang ([Z] p. 196) has proposed the following

Conjecture 2: *For any cyclic n-gon Π_n, we have*

$$\hat{L}_n^2 - 4(n \tan \frac{\pi}{n}) A_n \geq 0.$$

Equality holds if and only if Π_n is regular.

For any n-gon Π_n, we have the natural inequality $\hat{L}_n \leq L_n$. The equality $\hat{L}_n = L_n$ holds if and only if Π_n is regular (see Lemma (4-6) of [Ch]).

Moreover, it has been remarked by Zhang that Conjecture 2 implies Conjecture (2-6) of [K,K,Z] concerning the 3-parameter family of pseudo-perimeters denoted by $\mathcal{L}_n[x, (n-1)y, \frac{nz}{2}]$ for any n-gon inscribed in a circle. They prove that $\hat{L}_n \leq \mathcal{L}_n \leq L_n$ where $\mathcal{L}_n[1,0,0] = L_n$ and $\mathcal{L}_n[0,0,1] = \hat{L}_n$.

More generally, we also examine the following

Problem 2': *Let us consider a piecewise smooth closed curve $\mathcal{C}$ in the euclidean plane, of length L and area A. Let $(\Pi_n)_n$ be a sequence of n-gons approaching $\mathcal{C}$. L_n, $\hat{L}_n$ and A_n are respectively the perimeter, the pseudo-perimeter and the area of Π_n. Supposing that $\hat{L} = \lim_{n\to\infty} \hat{L}_n$ exists, do we have the Bonnesen-type inequality*

$$\hat{L}^2 - 4\pi A \geq 0 \ ?$$

Examples given below show that Problem 2' may have a solution.

In this paper, we shall discuss these conjectures and exhibit the special role played by $\tau_n = \frac{\varphi_n}{{\varphi_n}^0}$, where φ_n and ${\varphi_n}^0$ are defined as above for any cyclic n-gon, with sides $a_1, a_2, ..., a_n$.

Accordingly, we also introduce the ratio

$$\nu_n = (\frac{L_n}{\hat{L}_n})^{\frac{n}{2}-2}. \tag{11}$$

We will describe some examples. As a consequence we propose a conjecture which seems to be more appropriate than Conjecture 2. In particular, it yields bounds for τ_n. The theorem below shows that the position of τ_n compared with 1 and ν_n gives partial answers to both the above conjectures.

Theorem 1:
Let $\tau_n = \frac{\varphi_n}{{\varphi_n}^0}$ *and* $\nu_n = (\frac{L_n}{\hat{L}_n})^{\frac{n}{2}-2}$ *be the constants associated to any cyclic n-gon* Π_n *, with sides* $a_1, a_2, ..., a_n$. L_n *and* $\hat{L}_n$ *are respectively the perimeter and the pseudo-perimeter. We then have*
*(*i*) The inequality* $\tau_n \leq 1$ *implies conjecture* 1 *b) and conjecture* 2*. Moreover, this implication is strict.*
*(*ii*) The inequalities* $1 \leq \tau_n \leq \nu_n$ *imply conjecture* 1 *a) and conjecture* 2*.*
*(*iii*) The inequality* $\nu_n < \tau_n$ *contradicts conjecture* 2*.*
In these three cases, Equality $1 = \tau_n = \nu_n$ *holds if and only if* Π_n *is regular .*

Case (i) of Theorem 1 implies in particular that

$$\frac{\varphi_n}{{\varphi_n}^0} \leq 1 \leq (\frac{L_n}{\hat{L}_n})^{\frac{n}{2}-2}.$$

Case (ii) will be illustrated below by several examples. We hope that the following hypothesis $\varphi_n \leq {\varphi_n}^0$ will be verified by an n-gon.
As a corollary, we deduce from (ii) and (iii) that $\tau_n \leq \nu_n$ is equivalent to Conjecture 2.

Consequently, we also obtain the following result.

Corollary 2
Suppose $\tau_n \leq 1$ *is verified by a cyclic n-gon; we then have the following Bonnesen-type isoperimetric inequality :*

$$\hat{L}_n^2 - 4(n \tan\frac{\pi}{n})A_n \;\geq\; \hat{L}_n^2(1 - \frac{\varphi_n}{{\varphi_n}^0}).$$

Equality holds if and only if Π_n *is regular (i.e.* $\varphi_n = {\varphi_n}^0$*).*
Moreover, this inequality implies Conjecture 2 *.*

3 Proofs

1. Let L_n, $\hat{L}_n$, A_n be respectively the perimeter, pseudo-perimeter and area of any polygon Π_n as defined in the preceding section. The sides are of lengths $a_1, a_2, ..., a_n$. Consider ratio $\varphi_n = \dfrac{A_n}{P_n}$, where

$$P_n = \frac{L_n^2}{4}\sqrt{(1-\frac{2a_1}{L_n})(1-\frac{2a_2}{L_n}).....(1-\frac{2a_n}{L_n})} = \frac{1}{4}(\frac{n-2}{n})^{\frac{n}{2}}(\hat{L}_n)^{\frac{n}{2}}(L_n)^{\frac{4-n}{2}}. \tag{12}$$

Then we get expression

$$\varphi_n = (\frac{n-2}{n})^{-\frac{n}{2}}\ \frac{4A_n}{L_n^2}\ (\frac{L_n}{\hat{L}_n})^{\frac{n}{2}}.$$

After simplification, we have

$$\frac{\varphi_n}{{\varphi_n}^0} = \frac{4(n\tan\frac{\pi}{n})A_n}{L_n^2}\ (\frac{L_n}{\hat{L}_n})^{\frac{n}{2}}.$$

Consequently, we obtain a relation between τ_n and ν_n :

$$\tau_n = \frac{4(n\tan\frac{\pi}{n})A_n}{L_n^2}\ (\nu_n)^{\frac{n}{n-4}} \tag{13}$$

or

$$\tau_n = \frac{4(n\tan\frac{\pi}{n})A_n}{\hat{L}_n^2}\ \nu_n. \tag{14}$$

This proves that Conjecture 2 is equivalent to the inequality

$$\tau_n \leq \nu_n.$$

Furthermore, since examples given below verify condition (ii) of Theorem 1, $1 \leq \tau_n \leq \nu_n$, we may deduce that the implication (i) is necessarily strict.

Moreover, $\varphi_n \leq {\varphi_n}^0 \leq 1$ implies that $(\frac{L_n}{\hat{L}_n})^{\frac{n}{2}} \leq \dfrac{{L_n}^2}{4(n\tan\frac{\pi}{n})A_n}$. The latter implies

$$\nu_n \leq \frac{\hat{L}_n^2}{4(n\tan\frac{\pi}{n})A_n},$$

which is equivalent to Conjecture 2, since $\nu_n \geq 1$.

We may deduce from the above some necessary conditions satisfied by τ_n. Indeed, from (13) and (14), the ratio should verify the inequalities

$$\tau_n \leq {\nu_n}^{\frac{n}{n-4}}, \qquad \tau_n \geq \frac{4(n\tan\frac{\pi}{n})A_n}{\hat{L}_n^2} \geq \frac{4(n\tan\frac{\pi}{n})A_n}{L_n^2}.$$

All the equalities are attained only if $\nu_n = 1$, which corresponds to the regular polygon. Theorem 1 is proved.

2. We prove now Corollary 2. Since $\nu_n \geq 1$ we may deduce from (14) the following :

$$\frac{\varphi_n}{{\varphi_n}^0} \geq \frac{4(n\tan\frac{\pi}{n})A_n}{\hat{L}_n^2}.$$

We then get

$$1 - \frac{4(n\tan\frac{\pi}{n})A_n}{\hat{L}_n^2} \geq 1 - \frac{\varphi_n}{{\varphi_n}^0} \geq 0.$$

Thus,

$$0 \leq \hat{L}_n^2(1-\tau_n) \leq \hat{L}_n^2 - 4(n\tan\frac{\pi}{n})A_n.$$

So, we have proved the first part of Corollary 2. This inequality implies obviously $\hat{L}_n^2 - 4n\tan\frac{\pi}{n}A_n \geq 0$, i.e. Conjecture 2. Moreover, it is clear that equality is attained for the regular polygon Π_n.

Conversely, suppose we have

$$\hat{L}_n^2 - 4n\tan\frac{\pi}{n}A_n \ = \ \hat{L}_n^2(1 - \frac{\varphi_n}{{\varphi_n}^0}).$$

This is equivalent to

$$\tau_n = \frac{4n\tan\frac{\pi}{n}A_n}{L_n^2}. \tag{15}$$

That means $\quad \nu_n = 1, \quad$ i.e. Π_n is regular (see [Ch], Lemma(4.6)).

Remark

Under the hypothesis of Corollary 2, suppose in addition, that $\tau_n\nu_n \geq 1$. We then obtain a better Bonnesen-type isoperimetric inequality

$$\hat{L}_n^2(1-\tau_n) \leq \hat{L}_n^2(1-\frac{1}{\nu_n}) \leq \hat{L}_n^2 - 4n\tan\frac{\pi}{n}A_n.$$

4 Some special polygons

In this part, we shall see that Hypothesis (ii) of Theorem 1, which implies Conjecture 2, is in fact verified by many examples.

4.1 Example 1

It is true in particular for the Macnab polygon, which is a cyclic equiangular alternate-sided $2n$-gon with n sides of length a and n sides of length b. This polygon was originally used as an example by [K,K,Z] and by [Z], to test their conjectures.

In fact, we can do better by the following result:

Proposition 1

Let $\Pi_{n,n}$ be a cyclic $2n$-gon with n sides of length a alternatively with n sides of length b and $\varphi_{n,n}$ its associated function. Then, we have

$$1 \leq \frac{\varphi_{n,n}}{\varphi^0} \leq (\frac{L_{n,n}}{\hat{L}_{n,n}})^{\frac{n}{2}-2}.$$

Proof

A direct calculation gives the expression

$$\varphi_{n,n} = \frac{[(a^2+b^2)\cos\frac{\pi}{n} + 2ab]}{n\sin\frac{\pi}{n}(a+b)^2[1-\frac{2}{n}+\frac{4ab}{n^2(a+b)^2}]^{\frac{n}{2}}}.$$

This follows from expressions for $A_{n,n}$ and $\hat{L}_{n,n}$ calculated by [Z]. Indeed, one gets

$$A_{n,n} = \frac{n}{4\sin\frac{\pi}{n}}[(a^2+b^2)\cos\frac{\pi}{n} + 2ab],$$

$$\hat{L}^2_{n,n} = \frac{n^2}{(n-1)^2}[n(n-2)(a^2+b^2) + n^2ab + (n-2)^2ab].$$

Furthermore, it is easy to see that $\varphi_{n,n}$ may be written

$$\varphi_{n,n} = \frac{[\cos\frac{\pi}{n} + \frac{2ab}{(a+b)^2}(1-\cos\frac{\pi}{n}]}{n\sin\frac{\pi}{n}[1-\frac{2}{n}+\frac{4ab}{n^2(a+b)^2}]^{\frac{n}{2}}}.$$

Thus,

$$\frac{\varphi_{n,n}}{\varphi^0} = \frac{(1-\frac{1}{n})^n[\cos\frac{\pi}{n} + \frac{2ab}{(a+b)^2}(1-\cos\frac{\pi}{n}]}{1-\frac{2}{n}+\frac{4ab}{n^2(a+b)^2}]^{\frac{n}{2}}(1+\cos\frac{\pi}{n})},$$

which can be expressed as follows :

$$\frac{\varphi_{n,n}}{\varphi^0} = (1+\frac{E-1}{n(n-1)})^{-\frac{n}{2}}[1+(E-1)\frac{1-\cos\frac{\pi}{n}}{1+\cos\frac{\pi}{n}}], \text{ where } E = \frac{4ab}{(a+b)^2}.$$

We can see easily that $\frac{\varphi_{n,n}}{\varphi^0} \geq 1$, since $0 \leq E \leq 1$.

4.2 Example 2

Let Π_n^0 denote the regular n-gon whose sides a_i^0 are subtended by angles $\frac{\pi}{n}; i = 1, ..., n$. Consider a polygon Π_n^ε obtained from Π_n^0 by variations of a_1, a_2 which are subtended respectively by $\frac{\pi}{n} - \varepsilon$ and $\frac{\pi}{n} + \varepsilon$. The other sides of length $a_i^0 (3 \leq i \leq n)$ are unchanged. We prove that hypothesis (ii) is verified by Π_n^ε.

Proposition 2

Let Π_n^ε be the n-gon defined above for $n \geq 4$, φ_n^ε being its associated function. Then, for $\varepsilon > 0$ small, we have $1 \leq \frac{\varphi_n^\varepsilon}{\varphi_n^0} \leq (\frac{L_{n,n}}{\hat{L}_{n,n}})^{\frac{n}{2}-2}$.

Thus, it seems that the function $\varphi(a_{1,} a_{2,}...., a_n)$ for an n-gon possesses a local minimum for the regular polygons.

Proof

Let $L_n^\varepsilon, \hat{L}_n^\varepsilon, A_n^\varepsilon$ be respectively perimeter, pseudo-perimeter and enclosing area of the polygon Π_n^ε defined above. We get $a_1 = 2R\sin(\frac{\pi}{n} - \varepsilon)$ and $a_2 = 2R\sin(\frac{\pi}{n} + \varepsilon)$. After calculation, we obtain the following expression

$$A_n^\varepsilon = A_n^0(1 - \frac{4\varepsilon^2}{n}) \quad \text{and} \quad L_n^\varepsilon = L_n^0(1 - \frac{\varepsilon^2}{n}).$$

On the other hand,

$$\begin{aligned}(1 - \frac{2a_1}{L_n^\varepsilon})(1 - \frac{2a_2}{L_n^\varepsilon}) &= 1 - \frac{4}{n}[1 - \varepsilon^2(\frac{1}{2} - \frac{1}{n})] + \frac{4R^2[\sin^2\frac{\pi}{n} - \varepsilon^2)}{(L_n^0)^2(1 - \frac{\varepsilon^2}{n})} \\ &= (1 - \frac{2}{n})^2[1 + \frac{4\varepsilon^2}{(n-2)^2}(\frac{n-2}{2} + \frac{2}{n} - \frac{1}{\sin^2\frac{\pi}{n}})].\end{aligned}$$

Also, we get $(1 - \frac{2a_i^0}{L_n^\varepsilon}) = (1 - \frac{2}{n})(1 - \frac{2\varepsilon^2}{n(n-2)})$.
After simplification, we find the expression

$$\frac{\varphi_n^\varepsilon}{\varphi_n^0} = 1 - \frac{2\varepsilon^2}{(n-2)^2}[\frac{(n-2)^2}{2n} + \frac{n-2}{2} + \frac{2}{n} - \frac{1}{\sin^2\frac{\pi}{n}}],$$

which verifies $\frac{\varphi_n^\varepsilon}{\varphi_n^0} \geq 1$.
Notice that the factor ε^2 vanishes for $n = 4$.

From the expression

$$(\frac{L_{n,n}}{\hat{L}_{n,n}})^{\frac{n}{2}-2} = 1 - \frac{2(n-4)}{n(n-2)^2}\varepsilon^2[-\frac{(n-2)^2}{2n} + \frac{n-2}{2} + \frac{2}{n} - \frac{1}{\sin^2\frac{\pi}{n}}],$$

we also prove that $\frac{\varphi_n^\varepsilon}{\varphi_n^0} \leq (\frac{L_{n,n}}{\hat{L}_{n,n}})^{\frac{n}{2}-2}$.

5 Levy's polygons

In this part, we discuss the connexion between Conjecture 1 and some Bonnesen-type inequalities by using examples. Some n-gons satisfy Conjecture 1 without being regular. P. Levy has remarked on particular properties of the function φ_n which depends on the lengths of the sides

$$\varphi_n = \varphi_n(a_1, a_2, a_3, ..., a_n).$$

Indeed, he noticed that φ_n is a bounded algebraic symmetric function. Its bounds does not depend on n and it should verify the equality

$$\varphi_n(a_1, a_2, a_3, ..., 0) = \varphi_{n-1}(a_1, a_2, a_3, ..., a_{n-1}).$$

Consequently, we deduce that

$$\tau_{n-1}(a_1, a_2, ..., a_{n-1}) < \tau_n(a_1, a_2, ..., a_{n-1}, 0).$$

5.1 Also, P. Levy tried to find these bounds and tested Conjecture 1 on a special curve polygon denoted by $\Pi(\alpha)$, inscribed in the euclidean circle of radius 1. It is bounded by a circular arc with length $2(\pi-\alpha)$, and a chord of length $l = 2\sin\alpha$, where $0 \leq \alpha \leq \pi$.
$\Pi(\alpha)$ can be considered as limit of an $(n+1)$-gon with n sides of length $2\sin\frac{2\pi}{n}$ while only one has a fixed length $l = 2\sin\alpha$. Let $\varphi_n(\alpha)$ be the corresponding ratio and $\varphi(\alpha)$ its limit value when n tends to infinity. In this case, ${\varphi^0}_\infty = \frac{e}{\pi}$ is the limit value of ${\varphi^0}_n = \frac{1}{n\tan\frac{\pi}{n}\ (1-\frac{2}{n})^{n/2}}$.

We get the following

Proposition 3
Let $L(\alpha)$, $\hat{L}\alpha)$, $A(\alpha)$ be respectively the perimeter, the pseudo-perimeter and the enclosing area of the "polygon" $\Pi(\alpha)$, with $0 \leq \alpha \leq \pi$. We then obtain the inequalities
a) $1 \leq \frac{\varphi(\alpha)}{\varphi^0} \leq \frac{\pi}{e}\sqrt{\frac{e}{3}}$ *with* $\varphi(\pi) = \frac{e}{\pi}$ *and* $\varphi(0) = \sqrt{\frac{e}{3}}$.
b) $\hat{L}^2(\alpha) - 4\pi A(\alpha) \geq 0$.
Equality holds if and only if $\alpha = 0$.

Thus, we may deduce that $\Pi(\alpha)$ verifies Conjecture 1 and Problem 2'.

Proof
We may calculate the exact value of the function $\varphi(\alpha)$. We refer for that to P. Levy's papers [L] and [Ch] for details. Here $L = 2(\alpha + \sin\alpha)$ and $A = \alpha - \sin\alpha\cos\alpha$ so that

$$\varphi(\alpha) = \frac{(\pi - \alpha - \sin\alpha\cos\alpha)}{(\pi - \alpha + \sin\alpha)^{\frac{3}{2}}\sqrt{\pi - \alpha - \sin\alpha}} e^{\frac{\pi - \alpha}{\pi - \alpha + \sin\alpha}}.$$

$$\tau = \frac{\pi(\pi - \alpha - \sin\alpha\cos\alpha) e^{\frac{\sin\alpha}{\pi - \alpha + \sin\alpha}}}{(\pi - \alpha + \sin\alpha)^{\frac{3}{2}}\sqrt{\pi - \alpha - \sin\alpha}},$$

and

$$\nu = \sqrt{\frac{(\pi - \alpha + \frac{1}{2}\sin\alpha)}{(\pi - \alpha + \sin\alpha)}} e^{\frac{\pi - 2\alpha - \sin\alpha}{\alpha + \sin\alpha}}.$$

Thus, for $0 \leq \alpha \leq \pi$ we obtain the double inequality ([Ch], Proposition(2.1))

$$\frac{e}{\pi} \leq \varphi(\alpha) \leq \sqrt{\frac{e}{3}}.$$

These inequalities may also be verified by *Mathematica*. On the other hand, we may also deduce the expression $\frac{4\pi A}{\hat{L}^2}$ in terms of α :

$$\frac{4A}{\hat{L}^2} = \frac{(\pi - \alpha - \frac{1}{2}\sin 2\alpha)}{(\pi - \alpha + \sin\alpha)^2}.$$

We can prove easily that the right side of the above expression is a decreasing function of α, and for $\alpha = 0$, its value is 1. We then obtain part b) of Proposition 3.

5.2 P. Levy considered also another curvilinear polygon. Denote by $\Pi(\alpha, \theta)$ the polygon obtained from $\Pi(\alpha)$ by replacing the side with length $l = 2\sin\alpha$ by two sides. One of them has a length $2\sin\theta$. Then we get the expression of the perimeter and the area of the new polygon $\Pi(\alpha, \theta)$:

$$\begin{aligned} L(\alpha, \theta) &= 2[\pi - \alpha + \sin\theta + \sin(\alpha + \theta)], \\ A(\pi - \alpha, \theta) &= \pi - \alpha + \sin\alpha\cos(\alpha + 2\theta), \\ &\qquad\qquad\qquad 0 \leq \theta \leq \alpha. \end{aligned}$$

For $\theta = 0$ we get of course, $\Pi(\alpha, 0) \equiv \Pi(\alpha)$.

Proposition 4

Let $L(\alpha, \theta)$, $\hat{L}(\alpha, \theta)$, $A(\alpha, \theta)$ be respectively the perimeter, the pseudo-perimeter and the enclosing area of the "polygon" $\Pi(\alpha, \theta)$, with $0 \leq \alpha \leq \pi$, and $0 \leq \theta \leq \pi - \alpha$. We then obtain the inequalities

a) $\varphi(\alpha, \theta_0) \leq \varphi(\alpha, \theta) \leq \varphi(\alpha, \frac{\pi-\alpha}{2}) \leq 1$ for certain $\theta_0 > 0$.

b) $1 \leq \frac{\varphi(\alpha, \frac{\pi-\alpha}{2})}{\varphi^0} \leq \frac{\pi}{e}$ with $\varphi(\pi, 0) = 1$ and $\varphi(0, \frac{\pi}{2}) = \frac{\pi}{e}$.

c) $\hat{L}^2(\alpha, \frac{\pi-\alpha}{2}) - 4\pi A(\alpha, \frac{\pi-\alpha}{2}) \geq 0$.

Equality holds if and only if $\alpha = \pi$.

Proof

We calculate the following expression for the function $\varphi(\alpha, \theta)$ defined above

$$\varphi(\alpha, \theta) = \frac{\alpha - \sin\alpha\cos(\alpha + 2\theta)}{[\alpha + \sin\theta + \sin(\alpha + \theta)]\sqrt{\alpha^2 - 4\sin^2\frac{\alpha}{2}\cos^2(\frac{\alpha}{2} + \theta)}} e^{\frac{\alpha}{[\alpha + \sin\theta + \sin(\alpha + \theta)]}}$$

The details are given in [L] and [Ch]. In particular, for $0 \leq \alpha \leq \pi$ we have seen ([Ch], proposition (3-1)) that $\varphi(\alpha, \theta)$ admits a maximum $\theta_0 = \frac{\pi-\alpha}{2}$ and

two minima θ_1, θ_2 symmetric with respect to θ_0, such that $\varphi(\alpha, \theta_1) = \varphi(\alpha, \theta_2)$. Moreover, we may prove that $\frac{\varphi(\alpha, \frac{\pi-\alpha}{2})}{\varphi^0}$ is a decreasing function, $\varphi(\pi, 0) = 1$, and $\varphi(0, \frac{\pi}{2}) = \frac{\pi}{e}$.

Furthermore, after simplifying the expression $\frac{4\pi A(\alpha, \frac{\pi-\alpha}{2})}{\hat{L}^2(\alpha, \frac{\pi-\alpha}{2})}$ we find the following :

$$4\frac{A(\alpha, \frac{\pi-\alpha}{2})}{\hat{L}^2(\alpha, \frac{\pi-\alpha}{2})} = \frac{\pi - \alpha + \sin\alpha}{(\pi - \alpha + 2\cos\frac{\alpha}{2})^2}.$$

We may verify that a such function is decreasing and is less than 1. We have thus proved part c) of Proposition 4.

Remark : There are two possibilities for the "polygon" $\Pi(\alpha)$ (considered as a limit of an $(n+1)$-gon) with only one side of length $l = 2\sin\alpha$. The center of the circumscribed circle is inside the domain bounded by the $(n+1)$-gons. In this case, $\sum 2\theta_i = 2\pi$, where $2\theta_i$ is the subtended angle of the side a_i. The second case is arises by transposing α with $\pi - \alpha$. So, the center of the circomscribed circle is outside the domain bounded by the $(n+1)$-gons, and we have $\sum 2\theta_i < 2\pi$. This fact have been underlined by P.Levy. Of course, it is only in the first case, that the isoperimetric inequality is optimal.

6 Concluding remark

Thus, it is natural to expect that the hypothesis (ii) of Theorem 1 is verified for any cyclic n-gon. We then may propose the following

Conjecture 3 : *For any n-gon Π_n, we have the inequalities*

$$1 \leq \tau_n \leq \nu_n,$$

with $1 = \tau_n = \nu_n$ if and only if Π_n, is regular.

Obviously, this implies Conjecture 2 and Conjecture 1 a). Thus, conjecture 3 appears to be more significant than the previous conjectures. Notice that by Theorem 1,

$$\nu_n = 1 \Rightarrow \tau_n = 1.$$

To investigate in this direction, we can see for example expression (13). We may deduce anyway that $\dfrac{{\nu_n}^{\frac{n}{n-4}}}{\tau_n} \geq 1$, which is equivalent to

$$\frac{\varphi_n}{{\varphi_n}^0}\left(\frac{\hat{L}_n}{L_n}\right)^{\frac{n}{2}} \leq 1 \text{ and } \frac{1}{\nu_n^{\frac{4}{n-4}}} \leq \frac{\nu_n}{\tau_n}.$$

Equality holds if and only if the n-gon is regular. This gives an upper bound for τ_n.
Actually, by using the Bonnesen-style inequalities of X.M. Zhang [Z], we can improve it. More precisely, we have

$$\frac{\tau_n}{{\nu_n}^{\frac{2n}{n-4}}} \leq 1 - \left(\frac{2Rn\sin\frac{\pi}{n}}{L_n} - 1\right)^2.$$

Also,

$$\frac{\tau_n}{{\nu_n}^{\frac{2n}{n-4}}} \leq 1 - \left(1 - \frac{2rn\tan\frac{\pi}{n}}{L_n}\right)^2.$$

Here R and r are the circumradius and inradius, respectively.
Moreover, we get the following lower bound :

$$\left(\frac{2rn\tan\frac{\pi}{n}}{L_n}\right)^2 \leq \frac{\tau_n}{{\nu_n}^{\frac{2n}{n-4}}},$$

which implies in particular, that

$${\nu_n}^{\frac{n+4}{n-4}}\left(\frac{2rn\tan\frac{\pi}{n}}{L_n}\right)^2 \leq \frac{\tau_n}{\nu_n}.$$

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REFERENCES

[Ch] R. Chouikha *Probleme de P. Levy sur les polygones articules* C. R. Math. Report, Acad of Sc. of Canada, vol 10, p. 175-180, 1988.

[F] B. Fuglede *Bonnesen inequality for the isoperimetric deficiency of closed curves in the plane* Geometriae Dedicata, vol 38, p. 283-300, 1991.

[K.K.Z] H.T. Ku, M.C. Ku, X.M. Zhang *Analytic and geometric isoperimetric inequalities* J. of Geometry, vol 53, p.100-121, 1995.

[L] P. Levy *Le probleme des isoperimetres et des polygones articules* Bull. Sc. Math., 2eme serie, 90, p.103-112, 1966.

[O1] R. Osserman *The isoperimetric inequalities* Bull. Amer. Math. Soc., vol 84, p.1182-1238, 1978.

[O2] R. Osserman *Bonnesen-style isoperimetric inequalities* Amer. Math. Monthly, vol 1, p. 1-29, 1979.

[Z] X.M. Zhang *Bonnesen-style inequalities and pseudo-perimeters for polygons* J. of Geometry, vol 60, p.188-201, 1997.

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